The End-Periodic Index Theorem

Relation to the Atiyah-Patodi-Singer Index Theorem

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Abstract

Mrowka, Ruberman and Saveliev proved an index formula for end-periodic Dirac operators similar to that in the Atiyah-Singer (AS) index theorem. In my master thesis **[T]** the relation to the classical Atiyah-Patodi-Singer (APS) index theorem for manifolds with boundary is pointed out. Moreover, the proof of the end-periodic index theorem is given and its analogies to the heat equation proofs of the AS and the APS index theorem are illustrated.

Atiyah-Singer Index Theorem [AS63]

Theorem AS. *Let S be a graded Clifford bundle over a **closed** even-dimensional oriented Riemannian **manifold** X with induced Dirac operator $\mathcal{D}$. Then the positive chiral Dirac operator $\mathcal{D}^+\colon H^1(X,S^+)\to L^2(X,S^-)$ is Fredholm and its index is given by*

$$\boxed{\operatorname{ind}_{Fred}(\mathcal{D}^+)=\int_X \boldsymbol{I}(\mathcal{D})}\,,\qquad \text{where } \boldsymbol{I}(\mathcal{D}) \text{ denotes the local index form.}$$

Atiyah-Patodi-Singer Index Theorem [APS75]

Let $S\to Z$ be a graded Clifford bundle over a **compact** even-dimensional oriented Riemannian **manifold** with **boundary** Y and product structure in a collar of the boundary.

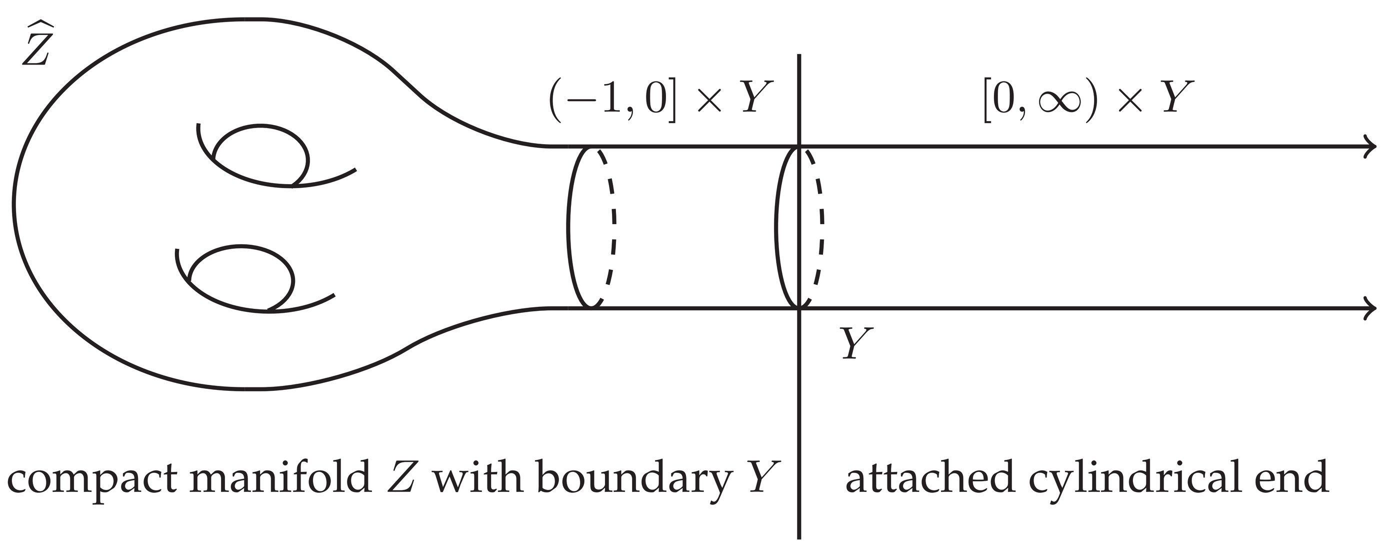
Assumption 1. *The boundary Dirac operator $\mathcal{D}(Y)\colon H^1(Y,S^+)\to L^2(Y,S^-)$ is invertible.*

Theorem APS1 (classical)**.** *The positive chiral Dirac operator $\mathcal{D}^+(Z)$ with domain restricted by the APS boundary condition is Fredholm. Its index is given by*

$$\boxed{\operatorname{ind}_{Fred}\left(\mathcal{D}^+(Z)|_{APS}\right)=\int_Z \mathbf{I}(\mathcal{D}(Z))-\frac{1}{2}\eta(\mathcal{D}(Y))}$$

with the η-invariant $\eta(\mathcal{D}(Y)):=\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^\infty t^{-1/2}\operatorname{Tr}\left(\mathcal{D}(Y)e^{-t\mathcal{D}(Y)^2}\right)dt.$

Define the open manifold $\widehat{Z}$ as $Z\cup_Y\left([0,\infty)\times Y\right)$ and extend all additional structures constantly through the cylindrical end.



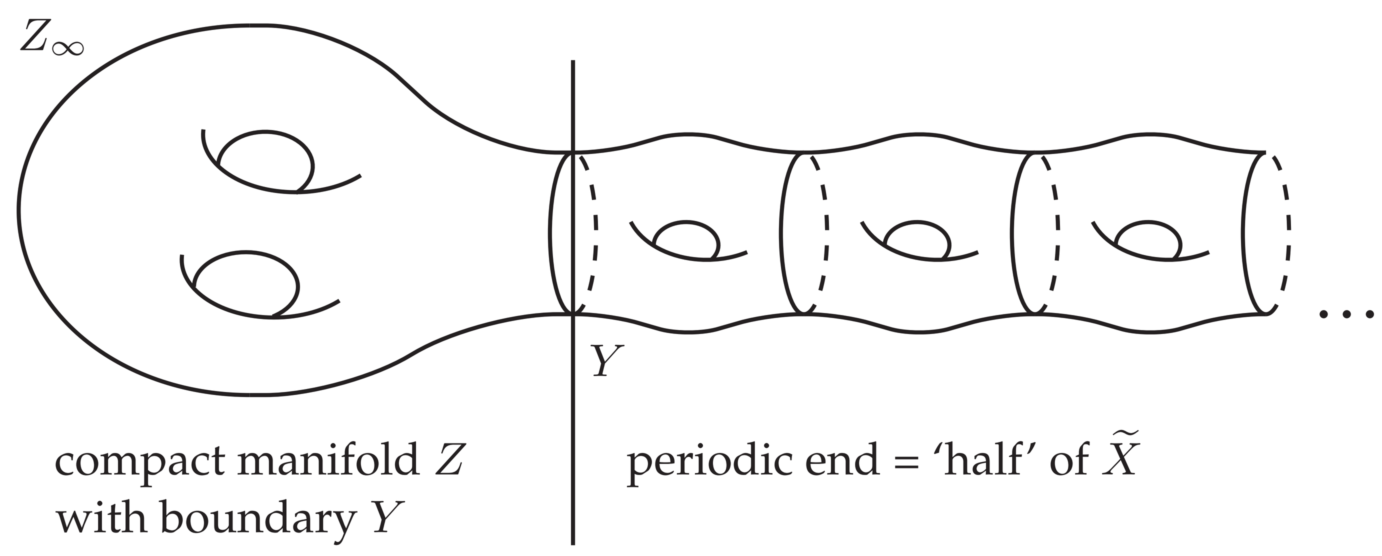
This leads to an index theorem for **manifolds** with **cylindrical end**:

Theorem APS2. *The positive chiral Dirac operator $\mathcal{D}^+(\widehat{Z})\colon H^1(\widehat{Z},S^+)\to L^2(\widehat{Z},S^-)$ is Fredholm and its index is given by*

$$\boxed{\operatorname{ind}_{Fred}(\mathcal{D}^+(\widehat{Z}))=\int_Z \mathbf{I}(\mathcal{D}(\widehat{Z}))-\frac{1}{2}\eta(\mathcal{D}(Y)).}$$

End-Periodic Index Theorem [MRS16]

Let Z_∞ be an even-dimensional **end-periodic manifold**, i.e. an oriented manifold that is built up by a compact manifold Z with boundary Y and an end modeled by 'half' of an infinite cylclic covering $\widetilde{X}$ over a compact oriented manifold X.



Further:
- $S\to Z_\infty$ is an end-periodic graded Clifford bundle.
- $\mathcal{D}^{(+)}(Z_\infty)$ and $\mathcal{D}^{(+)}(X)$ are the induced (chiral) Dirac operators.
- Fix a generator T of the group of covering translations of $\widetilde{X}$ and let $f\colon \widetilde{X}\to\mathbb{R}$ be the smooth map such that $f(Tx)=f(x)+1$ holds.

Assumption 2. *For a fixed branch of logarithm the twisted Dirac operator*

$$\mathcal{D}_z^+:=\mathcal{D}^+(X)-\ln z\cdot df\colon H^1(X,S^+)\to L^2(X,S^-)$$

is invertible for all $z\in\mathbb{C}$ with $|z|=1$.

Theorem MRS. *The positive chiral Dirac operator $\mathcal{D}^+(Z_\infty)\colon H^1(Z_\infty,S^+)\to L^2(Z_\infty,S^-)$ is Fredholm and its index is given by*

$$\boxed{\operatorname{ind}_{Fred}\left(\mathcal{D}^+(Z_\infty)\right)=\int_Z \boldsymbol{I}(\mathcal{D}(Z_\infty))-\int_Y\omega+\int_X df\wedge\omega-\frac{1}{2}\eta_{ep}(\mathcal{D}(X)).}$$

- $\eta_{ep}(\mathcal{D}(X)):=\dfrac{1}{i\pi}\displaystyle\int_0^\infty\int_{|z|=1}Tr\left(df\cdot D_z^+e^{-tD_z^-D_z^+}\right)\frac{dz}{z}dt.$
- ω *is an $(n-1)$ - form over X, such that $dw=\mathbf{I}(\mathcal{D}(X))$ holds.*

Relation between the Index Theorems

Relations.
- Theorem $APS1$ and Theorem $APS2$ are equivalent statements
- Theorem $APS2$ is a special case of the End-Periodic Index Theorem MRS

$\Longrightarrow$ The End-Periodic Index Theorem MRS generalizes the Index Theorem $APS1$

Remark. The classical APS index theorem is formulated using boundary conditions as in Theorem $APS1$ and it even holds, with a slight modification of the index formula, when the the boundary Dirac operator is not invertible. There exist also generalizations of Theorem $APS2$ and Theorem MRS using weighted Sobolev spaces, so that Assumption 1 respectivly Assumption 2 can be omitted and the previous relations still hold.

References

[APS75] M. F. Atiyah, V. K. Patodi, and I. M. Singer. "Spectral asymmetry and Riemannian geometry. I". English. In: *Math. Proc. Camb. Philos. Soc.* 77 (1975), pp. 43–69.

[AS63] M. F. Atiyah and I. M. Singer. "The index of elliptic operators on compact manifolds". English. In: *Bull. Am. Math. Soc.* 69 (1963), pp. 422–433.

[MRS16] T. Mrowka, D. Ruberman, and N. Saveliev. "An index theorem for end-periodic operators". English. In: *Compos. Math.* 152.2 (2016), pp. 399–444.